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A remark on the two-dimensional Jacobian conjecture

Arno van den Essen^{a,*}, Halszka Tutaj^b^a*Department of Mathematics, Catholic University Nijmegen, Nijmegen, The Netherlands*^b*Department of Mathematics, Jagiellonian University, Krakow, Poland*

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Introduction

Let k be an algebraically closed field of characteristic zero and $F: k^2 \rightarrow k^2$ a polynomial map. It is well-known that the Jacobian conjecture is equivalent to the statement that $\det JF \in k^*$ implies the injectivity of F .

In this paper we improve upon this result. More precisely we show the following: if $\det JF \in k^*$ and there exist a point $a \in k^2$ and three lines through a such that the restriction of F to each of these lines is injective, then F is invertible. As a consequence we show that each F with $\det JF \in k^*$ which is of the form $F = X + H$, where H is homogeneous of some degree d , is invertible.

1. Preliminaries

Throughout this paper k denotes an algebraically closed field of characteristic zero and $k[t]$ the polynomial ring in the variable t over k .

Let $f(t)$ and $g(t)$ be two non-constant polynomials in $k[t]$. In [4] the D -resultant of $f(t)$ and $g(t)$ is defined as

$$D\text{Res}_t(f(t), g(t)) = \text{Res}_t\left(\frac{f(t) - f(s)}{t - s}, \frac{g(t) - g(s)}{t - s}\right) \quad (s \text{ is a new variable})$$

So if we denote this D -resultant by $D(s)$ we have that $D(s) \in k[s]$. Furthermore it is proved in [4, Theorem 2.1] that $k(f(t), g(t)) = k(t)$ if and only if $D(s) \neq 0$ and $k[f(t), g(t)] = k[t]$ if and only if $D(s) \in k^*$. The following lemma plays a crucial role in the proof of our main theorem.

*Corresponding author.

Lemma 1.1. Let $f(t), g(t)$ in $k[t]$. If $(f'(t_0), g'(t_0)) \neq (0, 0)$ for all $t_0 \in k$ and if the map $k \ni t_0 \mapsto \gamma(t_0) = (f(t_0), g(t_0)) \in k^2$ is injective, then $D\text{Res}_t(f(t), g(t)) \in k^*$.

Proof. Suppose $D(s) \notin k^*$. Then there exists $s_0 \in k$ such that $D(s_0) = 0$. So the polynomials

$$\frac{f(t) - f(s_0)}{t - s_0} \quad \text{and} \quad \frac{g(t) - g(s_0)}{t - s_0}$$

have a common zero t_0 in k .

Case 1: $t_0 \neq s_0$. Then $f(t_0) = f(s_0)$ and $g(t_0) = g(s_0)$, implying that $\gamma(t_0) = \gamma(s_0)$ contradicting the injectivity of γ .

Case 2: $t_0 = s_0$. Then

$$\left. \frac{f(t) - f(s_0)}{t - s_0} \right|_{t=s_0} = \left. \frac{g(t) - g(s_0)}{t - s_0} \right|_{t=s_0} = 0.$$

Since these elements are equal to $f'(s_0)$ resp. $g'(s_0)$ (cf. [4, Lemma 0.7]), we get that $(f'(s_0), g'(s_0)) = (0, 0)$, contradicting the hypothesis.

Apparently $D(s) \in k^*$, as desired. \square

2. The main results

From now on let $F = (f, g): k^2 \rightarrow k^2$ be a polynomial map. For each vector $0 \neq v \in k^2$ consider the restriction of F to the line through v and the origin i.e. the map $k \ni t_0 \mapsto F_v(t_0) = (f(t_0 v), g(t_0 v)) \in k^2$.

Of course, if F is invertible then $\det JF \in k^*$ and $F_v: k \rightarrow k^2$ is injective for each $0 \neq v \in k^2$. Our main result, Theorem 2.1, gives a stronger converse of this result, namely:

Theorem 2.1. Let $\det JF \in k^*$. If there exist three lines through the origin, say spanned by the vectors $v^{(1)}, v^{(2)}$ and $v^{(3)}$, such that for each i the restriction $F_{v^{(i)}}$ of F to the line through $v^{(i)}$ and the origin is injective, then F is invertible.

Proof. Since $\det JF \in k^*$ there exists a homogeneous polynomial h such that $f = \lambda_1 h^r + \text{lower order terms}$ and $g = \lambda_2 h^s + \text{lower order terms}$, with λ_1, λ_2 in k^* and $r, s \geq 1$ (cf. [6, Theorem 1.4]). Furthermore, h has at most two irreducible factors i.e. $h = (\alpha_1 x + \beta_1 y)^p (\alpha_2 x + \beta_2 y)^q$ for some $\alpha_i, \beta_j \in k$ and $p, q \in \bar{\mathbb{N}}$ (cf. [1] and [5]). Now let $v \in k^2$. Since $\det JF \in k^*$ it follows from the chain rule that $F'_v(t_0) \neq (0, 0)$ for all $t_0 \in k$. So if $t_0 \mapsto F_v(t_0)$ is injective it follows from Lemma 1.1 that $k[f(tv), g(tv)] = k[t]$. Then the Abhyankar–Moh theorem (cf. [2]) implies that $\deg_t f(tv)$ divides $\deg_t g(tv)$ or $\deg_t g(tv)$ divides $\deg_t f(tv)$. From the form of h it follows that $h(v^{(i)}) \neq 0$ for at least one $1 \leq i \leq 3$. For this i we have that $\deg_t f(tv^{(i)}) = \deg f$ and $\deg_t g(tv^{(i)}) = \deg g$. So

we conclude that $\deg f$ divides $\deg g$ or $\deg g$ divides $\deg f$. Say $\deg g$ divides $\deg f$. Hence $r = ms$ for some $m \in \mathbb{N}$. Then the pair $(f_1, g_1) = (f - (\lambda_1/\lambda_2^m)g^m, g)$ satisfies $\deg f_1 < \deg f$. Furthermore, one easily verifies that $\det J(f_1, g_1) \in k^*$ and that (f_1, g_1) is injective on the three lines spanned by the $v^{(i)}$. The result follows by induction on $\deg f + \deg g$. \square

Corollary 2.2. *Let $F = X + H: k^2 \rightarrow k^2$ be a polynomial map, where H is homogeneous of degree $d \geq 2$. If $\det JF \in k^*$, then F is invertible.*

Proof. We may assume that $H \neq 0$. So there exist infinitely many elements $0 \neq v \in k^2$ with $H(v) \neq 0$.

(i) First we show that for such v the vectors v and $H(v)$ are linearly independent over k ; namely if $H(v) = \lambda v$ for some $\lambda \in k$, then $k(F_v(t)) = k(tv + t^d \lambda v) = k(t + \lambda t^d)$. Since $\det JF \in k^*$ we get, as before, that $F'_v(t_0) \neq (0, 0)$ for all t_0 in k . So by [3, Lemma 3.4] we have $k(F_v(t)) = k(t)$, implying that $k(t + \lambda t^d) = k(t)$. It follows that $\lambda = 0$ (for example by writing $k(t + \lambda t^d) = k(t + \lambda t^d, t + \lambda t^d)$ and using the D -resultant criterion for rational function fields). So $H(v) = 0$, a contradiction.

(ii) Finally if v and $H(v)$ are linearly independent over k , then F_v is injective: if $F_v(t) = F_v(s)$ then $F(tv) = F(sv)$, hence $tv + t^d H(v) = sv + s^d H(v)$, so $(t - s)v + (t^d - s^d)H(v) = 0$. Since v and $H(v)$ are linearly independent over k we conclude that $t - s = 0$, so $t = s$.

So by (i) and (ii) we obtain that the restriction of F to infinitely many lines through the origin is injective. Then apply Theorem 2.1. \square

Note added in proof. After receiving a preprint of this paper Janusz Gwoździwicz has improved upon our result: he shows that injectivity on one line is sufficient (Bull. Soc. Sci. Łódź XV (7) (1993) 59–60).

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Halszka Tutaj

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